INVERSE TRIGONOMETRIC FUNCTION

1. INTRODUCTION :

The inverse trigonometric functions, denoted by $\sin^{-1}x$ or (arc sinx), $\cos^{-1}x$ etc., denote the angles whose sine, cosine etc, is equal to x. The angles are usually the numerically smallest angles, except in the case of $\cot^{-1}x$, and if positive & negative angles have same numerical value, the positive angle has been chosen.

It is worthwhile noting that the functions sinx, cosx etc are in general not invertible. Their inverse is defined by choosing an appropriate domain & co-domain so that they become invertible. For this reason the chosen value is usually the simplest and easy to remember.

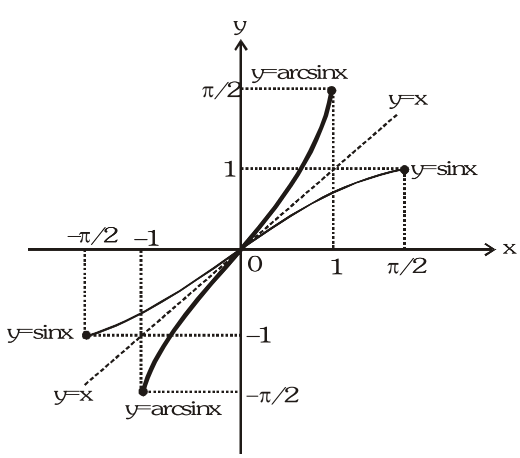
2. DOMAIN & RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS :

S.No	f (x)	Domain	Range
(1)	$\sin^{-1}x$	$\|x\| \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(2)	$\cos^{-1}x$	$\|x\| \le 1$	$[0, \pi]$
(3)	$\tan^{-1}x$	$x \in R$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(4)	$\sec^{-1}x$	$\|x\| \ge 1$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
(5)	$\text{cosec}^{-1}x$	$\|x\| \ge 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
(6)	$\cot^{-1}x$	$x \in R$	$(0, \pi)$

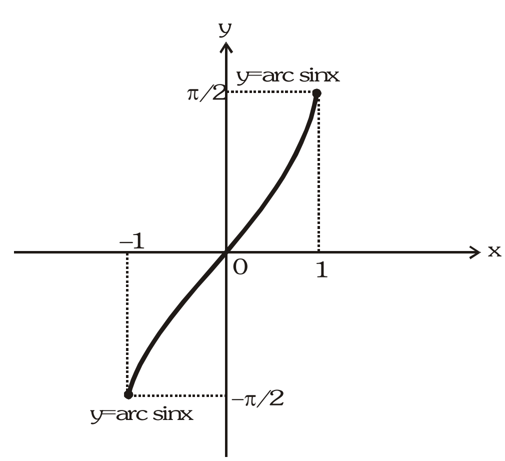
3. GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ $\quad$ $f^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$

$f(x) = \sin x$ $\quad$ $f^{-1}(x) = \sin^{-1}(x)$



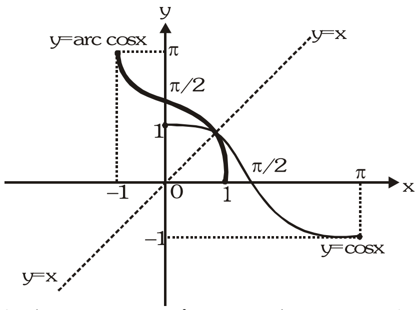
(Taking image of sin x about y = x to get $\sin^{-1}x$)



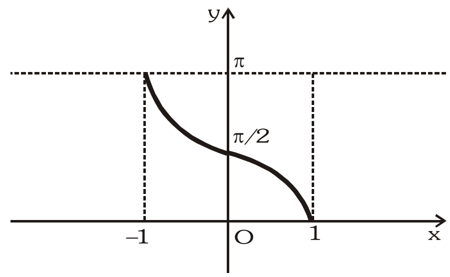
($y = \sin^{-1}x$)

(b) $f : [0, \pi] \to [-1, 1]$ $\quad$ $f^{-1} : [-1, 1] \to [0, \pi]$

$f(x) = \cos x$ $\quad$ $f^{-1}(x) = \cos^{-1} x$



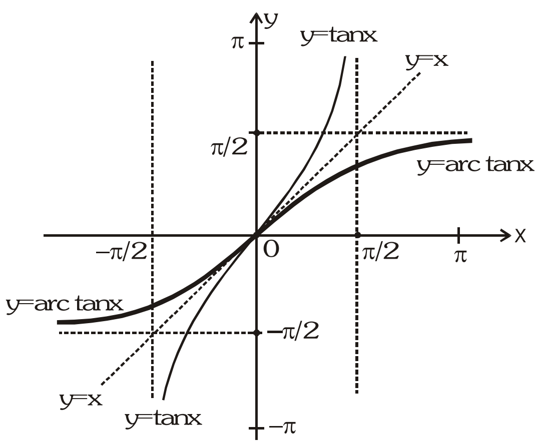
(Taking image of cos x about y = x)



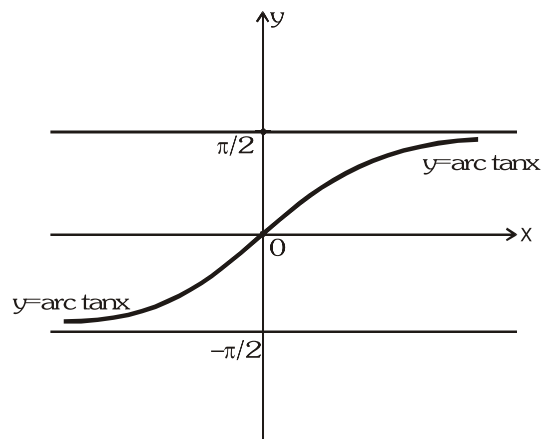
($y = \cos^{-1}x$)

(c) $f : (-\pi/2, \pi/2) \to R$ $\quad$ $f^{-1} : R \to (-\pi/2, \pi/2)$

$f(x) = \tan x$ $\quad$ $f^{-1}(x) = \tan^{-1} x$



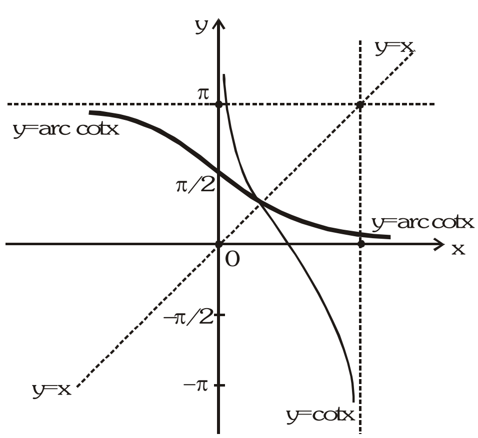
(Taking image of tan x about y = x)



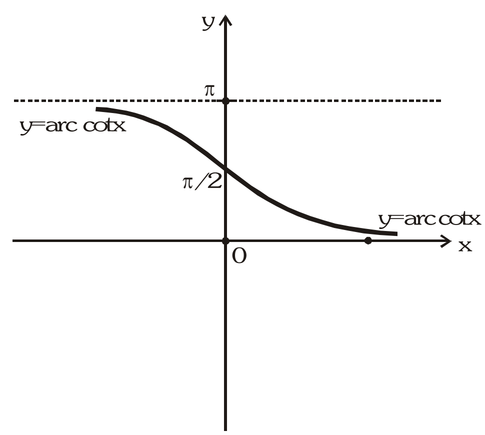
($y = \tan^{-1}x$)

(d) $f : (0, \pi) \to R$ $\quad$ $f^{-1} : R \to (0, \pi)$

$f(x) = \cot x$ $\quad$ $f^{-1}(x) = \cot^{-1} x$



(Taking image of cot x about y = x)



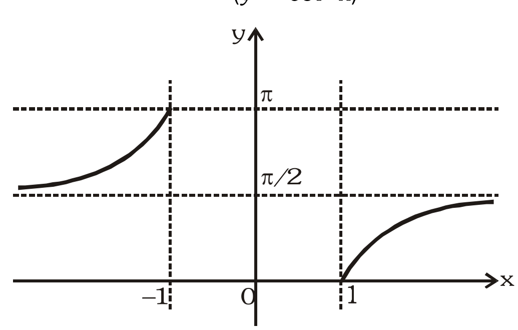
($y = \cot^{-1}x$)

(e) $f : [0, \pi/2) \cup (\pi/2, \pi] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \sec x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \to [0, \pi/2) \cup (\pi/2, \pi]$

$f^{-1}(x) = \sec^{-1} x$

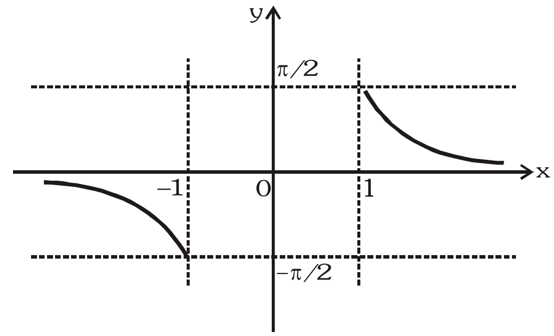


(f) $f : [-\pi/2, 0) \cup (0, \pi/2] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \text{cosec } x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \to [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \text{cosec}^{-1} x$



From the above discussions following IMPORTANT points can be concluded.

(i) All the inverse trigonometric functions represent an angle.

(ii) If $x \geq 0$, then all six inverse trigonometric functions viz $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$, $\cot^{-1}x$ represent an acute angle.

(iii) If $x < 0$, then $\sin^{-1}x$, $\tan^{-1}x$ & $\text{cosec}^{-1}x$ represent an angle from $-\pi/2$ to 0 (IV^{th} quadrant)

(iv) If $x < 0$, then $\cos^{-1} x$, $\cot^{-1}x$ & $\sec^{-1}x$ represent an obtuse angle. (II^{nd} quadrant)

(v) III^{rd} quadrant is never used in inverse trigonometric function.

Illustration 1 : The value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{5\pi}{12}$ (C) $\frac{3\pi}{4}$ (D) $\frac{13\pi}{12}$

Solution : $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$ **Ans.(C)**

Illustration 2 : If $\sum_{i=1}^{2n} \cos^{-1} x_i = 0$ then find the value of $\sum_{i=1}^{2n} x_i$

Solution : We know, $0 \leq \cos^{-1} x \leq \pi$

Hence, each value $\cos^{-1}x_1, \cos^{-1}x_2, \cos^{-1}x_3, \ldots\ldots, \cos^{-1}x_{2n}$ are non-negative their sum is zero only when each value is zero.

i.e., $\cos^{-1}x_i = 0$ for all i

$\Rightarrow$ $x_i = 1$ for all i

$\therefore$ $\sum_{i=1}^{2n} x_i = x_1 + x_2 + x_3 + \ldots\ldots + x_{2n} = \underbrace{\{1 + 1 + 1 \ldots\ldots\ldots + 1\}}_{2n \text{ times}} = 2n$ {using (i)}

$\Rightarrow$ $\sum_{i=1}^{2n} x_i = 2n$ **Ans.**

Do yourself - 1 :

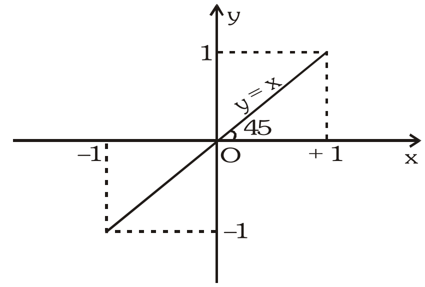
(i) If α, β are roots of the equation $6x^2 + 11x + 3 = 0$, then

(A) both $\cos^{-1}\alpha$ and $\cos^{-1}\beta$ are real (B) both $\text{cosec}^{-1}\alpha$ and $\text{cosec}^{-1}\beta$ are real

(C) both $\cot^{-1}\alpha$ and $\cot^{-1}\beta$ are real (D) none of these

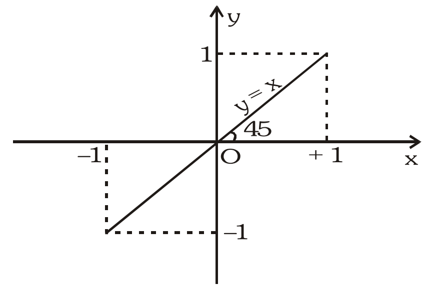
(ii) If $\sin^{-1}x + \sin^{-1}y = \pi$ and $x = ky$, then find the value of $39^{2k} + 5^k$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :



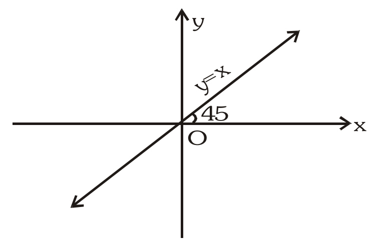
P-1 (i) $y = \sin(\sin^{-1}x) = x$

$x \in [-1,1],\ y \in [-1,1]$



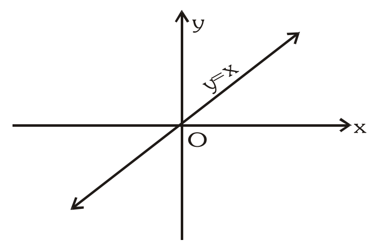
(ii) $y = \cos(\cos^{-1}x) = x$

$x \in [-1,1],\ y \in [-1,1]$



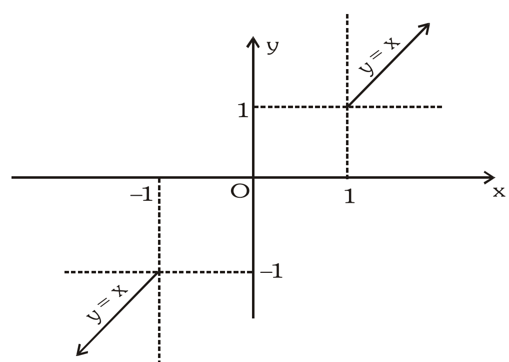
(iii) $y = \tan(\tan^{-1}x) = x$

$x \in R,\ y \in R$



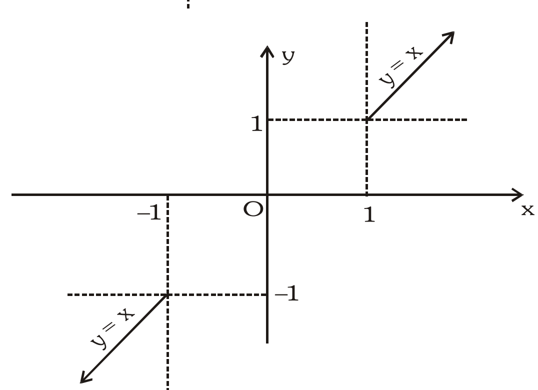
(iv) $y = \cot(\cot^{-1}x) = x,$

$x \in R;\ y \in R$



(v) $y = \text{cosec}(\text{cosec}^{-1}x) = x,$

$|x| \geq 1,\ |y| \geq 1$



(vi) $y = \sec(\sec^{-1}x) = x$

$|x| \geq 1\ ;\ |y| \geq 1$

Note : All the above functions are aperiodic.

Illustration 3 : Evaluate the following :

(i) $\sin(\cos^{-1}3/5)$ (ii) $\cos(\tan^{-1} 3/4)$ (iii) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution : (i) Let $\cos^{-1} 3/5 = \theta$. Then,

$\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$

$\therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$. Then,

$\tan\theta = 3/4$

$\Rightarrow \cos\theta = \frac{4}{5}$ $\left\{\because \text{ as } \cos^2\theta = \frac{1}{1+\tan^2\theta}\right\}$

$\therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{2}-\left(-\frac{\pi}{6}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ **Ans.**

Do yourself - 2 :

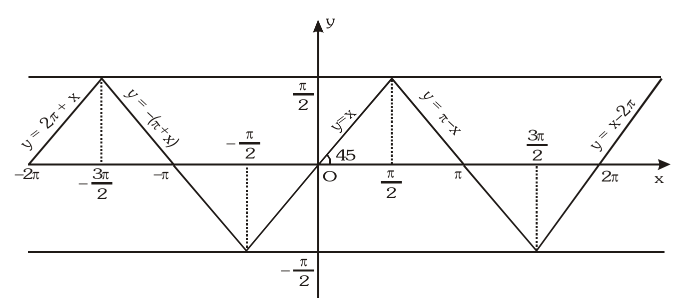
Evaluate the following :

(i) $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right)$ (ii) $\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right)$ (iii) $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

(iv) $\sin\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right)$ (v) $\cos\left(\sin^{-1}\frac{1}{2}\right)$ (vi) $\sin\left(\cos^{-1}\frac{3}{5}\right)$

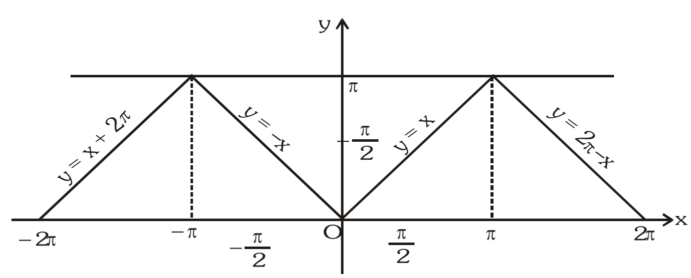
P-2 (i) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ periodic with period 2π and it is an odd function.

$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x, & -\pi \le x \le -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$$



(ii) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0,\pi]$, periodic with period 2π and it is an even function.

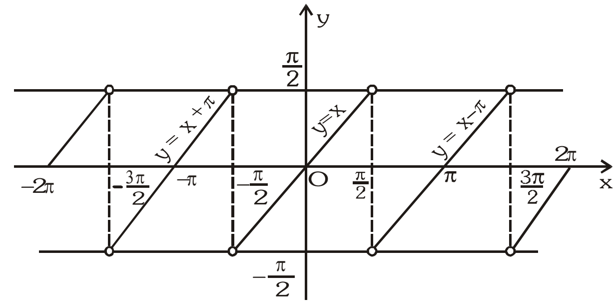
$$\cos^{-1}(\cos x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 < x \le \pi \end{cases}$$



(iii) $y = \tan^{-1}(\tan x)$

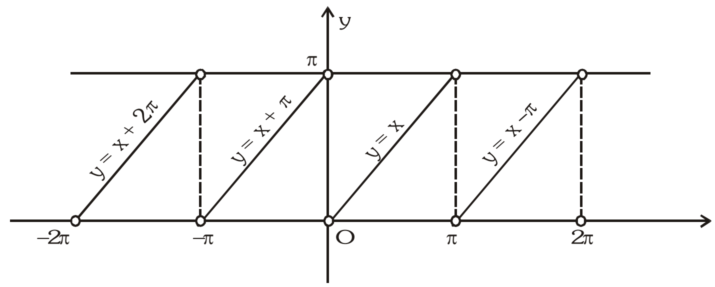
$x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}$; $y \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, periodic with period π and it is an odd function.

$$\tan^{-1}(\tan x) = \begin{cases} x+\pi \ , & -\dfrac{3\pi}{2} < x < -\dfrac{\pi}{2} \\ x \ , & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ x-\pi \ , & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$$

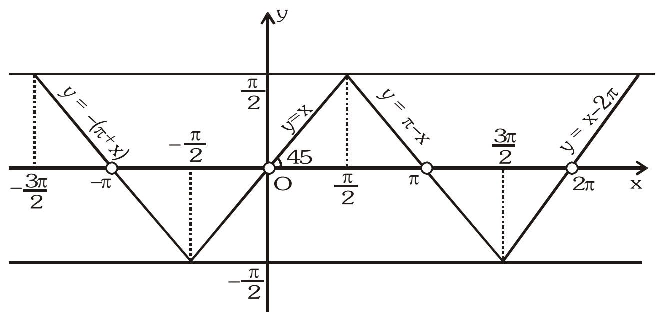


(iv) $y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi, n \in I\}$, $y \in (0, \pi)$, periodic with period π and neither even nor odd function.

$$\cot^{-1}(\cot x) = \begin{cases} x+\pi \ , & -\pi < x < 0 \\ x \ , & 0 < x < \pi \\ x-\pi \ , & \pi < x < 2\pi \end{cases}$$



(v) $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in R - \{n\pi, n \in I\}$ $y \in \left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$, is periodic with period 2π and it is an odd function.



(vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π and it is an even function.

$$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

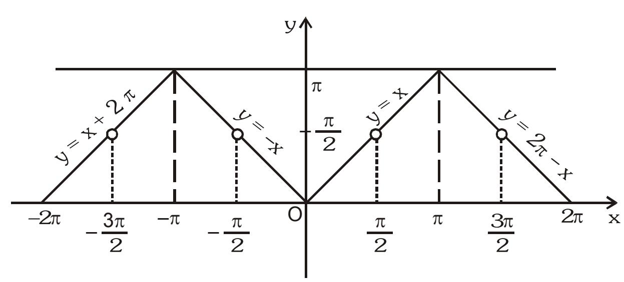


Illustration 4 : The value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$ is -

(A) $5\pi/6$ (B) $\pi/2$ (C) $3\pi/2$ (D) none of these

Solution : $\sin^{-1}\left(-\sqrt{3}/2\right) = -\sin^{-1}\left(\sqrt{3}/2\right) = -\pi/3$

and $\cos^{-1}(\cos(7\pi/6)) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos 7\pi/6) = -\dfrac{\pi}{3} + \dfrac{5\pi}{6} = \dfrac{\pi}{2}$ **Ans.(B)**

Illustration 5 : Evaluate the following :

(i) $\sin^{-1}(\sin\pi/4)$ (ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Solution : (i) $\sin^{-1}(\sin\pi/4) = \frac{\pi}{4}$

(ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right)$ $\left\{\because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}\right\}$

$= \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$ **Ans.**

Illustration 6 : Evaluate the following :

(i) $\sin^{-1}(\sin 10)$ (ii) $\tan^{-1}(\tan(-6))$ (iii) $\cot^{-1}(\cot 4)$

Solution : (i) We know that $\sin^{-1}(\sin\theta) = \theta$, if $-\pi/2 \le \theta \le \pi/2$

Here, $\theta = 10$ radians which does not lie between $-\pi/2$ and $\pi/2$

But, $3\pi - \theta$ i.e., $3\pi - 10$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Also, $\sin(3\pi - 10) = \sin 10$

$\therefore$ $\sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = (3\pi - 10)$

(ii) We know that,

$\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$. Here, $\theta = -6$, radians which does not lie between $-\pi/2$ and $\pi/2$. We find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that;

$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$

$\therefore$ $\tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = (2\pi - 6)$

(iii) $\cot^{-1}(\cot 4) = \cot^{-1}(\cot(\pi + (4 - \pi))) = \cot^{-1}(\cot(4 - \pi)) = (4 - \pi)$ **Ans.**

Illustration 7 : Prove that $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3) = 15$

Solution : We have,

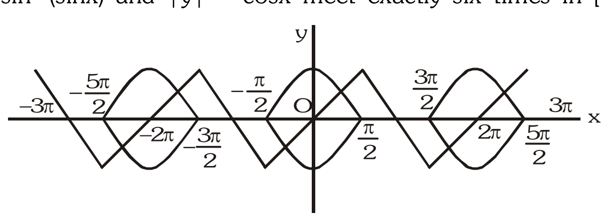
$\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3)$

$$= \left\{\sec\left(\tan^{-1}2\right)\right\}^2 + \left\{\text{cosec}\left(\cot^{-1}3\right)\right\}^2 = \left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\text{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$= \left\{\sec\left(\sec^{-1}\sqrt{5}\right)\right\}^2 + \left\{\text{cosec}\left(\text{cosec}^{-1}\sqrt{10}\right)\right\}^2 = \left(\sqrt{5}\right)^2 + \left(\sqrt{10}\right)^2 = 15$$

Illustration 8 : Find the number of solutions of (x, y) which satisfy $|y| = \cos x$ and $y = \sin^{-1}(\sin x)$, where $|x| \le 3\pi$.

Solution : Graphs of $y = \sin^{-1}(\sin x)$ and $|y| = \cos x$ meet exactly six times in $[-3\pi, 3\pi]$.



Do yourself - 3 :

Evaluate the following :

(i) $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ (ii) $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$ (iii) $\sin^{-1}(\sin 2)$ (iv) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

(v) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ (vi) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ (vii) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

P-3 (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ $\quad -1 \le x \le 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ $\quad x \in R$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$ $\quad |x| \ge 1$

P-4 (i) $\sin^{-1}(-x) = -\sin^{-1}x$, $\quad -1 \le x \le 1$

(ii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$, $x \le -1$ or $x \ge 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$, $\quad x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $\quad x \in R$

(v) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $\quad -1 \le x \le 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $\quad x \le -1$ or $x \ge 1$

P-5 (i) $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$; $\quad x \le -1,\ x \ge 1$

(ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$; $\quad x \le -1,\ x \ge 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x} & ;\ x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ;\ x < 0 \end{cases}$

Illustration 9 : Prove that $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \begin{cases} \pi/2 & ,\ \text{if } x > 0 \\ -\pi/2 & ,\ \text{if } x < 0 \end{cases}$

Solution : We have , $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x & ,\ \text{for } x > 0 \\ -\pi + \cot^{-1}x & ,\ \text{for } x < 0 \end{cases}$

$$\therefore \quad \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1}x + \cot^{-1}x = \pi/2 & ,\ \text{if } x > 0 \\ \tan^{-1}x + \cot^{-1}x - \pi = \pi/2 - \pi = -\pi/2 & ,\ \text{if } x < 0 \end{cases}$$

Do yourself - 4 :

(i) Prove the following :

(a) $\cos^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right)$ (b) $\sin^{-1}\left(-\frac{4}{5}\right) = \tan^{-1}\left(-\frac{4}{3}\right) = \cos^{-1}\left(-\frac{3}{5}\right) - \pi$

(ii) Find the value of $\sin(\tan^{-1}a + \tan^{-1}\frac{1}{a})$; $a \neq 0$

P-6 (i) (a) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy} & \text{where } x > 0,\ y > 0\ \&\ xy < 1 \\ \pi + \tan^{-1}\dfrac{x+y}{1-xy} & \text{where } x > 0,\ y > 0\ \&\ xy > 1 \\ \dfrac{\pi}{2}, & \text{where } x > 0,\ y > 0\ \&\ xy = 1 \end{cases}$

(b) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x-y}{1+xy}$ where $x > 0,\ y > 0$

(c) $\tan^{-1}x + \tan^{-1}y + \tan^{-1} z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if $x > 0,\ y > 0,\ z > 0\ \&\ xy + yz + zx < 1$

(ii) (a) $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] & \text{where } x > 0, y > 0\ \&\ (x^2 + y^2) \le 1 \\ \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] & \text{where } x > 0, y > 0\ \&\ x^2 + y^2 > 1 \end{cases}$

(b) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0,\ y > 0$

(iii) (a) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$ where $x > 0,\ y > 0$

(b) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ;\quad x < y,\quad x, y > 0 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ;\quad x > y,\quad x, y > 0 \end{cases}$

Note : In the above results x & y are taken positive. In case if these are given as negative, we first apply P-4 and then use above results.

Illustration 10 **:** Prove that

(i) $\tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{13} = \tan^{-1}\dfrac{2}{9}$ (ii) $\tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}$

Solution **:** (i) L.H.S. $= \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{13}$

$$= \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\} \qquad \left\{\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right);\ \text{if } xy < 1\right\}$$

$$= \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\left(\frac{2}{9}\right) = \text{R.H.S.}$$

(ii) $\left(\tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7}\right) + \left(\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) = \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Ans.

Illustration 11 : Prove that $\sin^{-1}\frac{12}{13}+\cot^{-1}\frac{4}{3}+\tan^{-1}\frac{63}{16}=\pi$

Solution : We have,

$$\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}$$

$$=\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16} \qquad \left[\because \sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5}\text{and }\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}\right]$$

$$=\pi+\tan^{-1}\left\{\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right\}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right),\text{if } xy>1\right]$$

$$=\pi+\tan^{-1}\left(\frac{63}{-16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}(-x)=-\tan^{-1}x\right]$$

$$=\pi$$

Illustration 12 : Prove that : $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Solution : We have, $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{3}{5}$ $\left[\because \cos^{-1}\frac{12}{13}=\sin^{-1}\frac{5}{13}\right]$

$$=\sin^{-1}\left\{\frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2}\right\}=\sin^{-1}\left\{\frac{5}{13}\times\frac{4}{5}+\frac{3}{5}\times\frac{12}{13}\right\}=\sin^{-1}\frac{56}{65}$$

Illustration 13 : If $x = \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a)))))$, where $a \in [0, 1]$. Find the relationship between x and y in terms of 'a'

Solution : Here, $x = \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ $\left\{ \text{Let } \sin\theta = a \Rightarrow \sec\theta = \frac{1}{\sqrt{1-a^2}}\right.$

$$= \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec\theta))))$$

$$\Rightarrow x = \text{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$$

$\left\{ \text{Let } \cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)=\phi \Rightarrow \cot\phi=\frac{1}{\sqrt{1-a^2}}\right.$

$$= \text{cosec}(\tan^{-1}(\cos\phi))$$

therefore $\cos\phi=\frac{1}{\sqrt{2-a^2}}$

$$\Rightarrow x = \text{cosec}\left(\tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right)$$

$\left\{ \text{Let, } \tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)=\psi \Rightarrow \tan\psi=\frac{1}{\sqrt{2-a^2}}\right.$

$$= \text{cosec}\,\psi$$

therefore $\text{cosec}\,\psi=\sqrt{3-a^2}$

$$\Rightarrow \quad x = \sqrt{3-a^2} \qquad \text{...... (i)}$$

and $\quad y = \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}(\cos^{-1} a)))))$ $\quad$ $\left\{ \text{Let } \cos^{-1}a = \alpha \Rightarrow \cos\alpha = a \Rightarrow \text{cosec}\alpha = \dfrac{1}{\sqrt{1-a^2}} \right.$

$= \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}\alpha))))$

$\Rightarrow \quad y = \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$ $\quad$ $\left\{ \text{Let, } \tan^{-1}\dfrac{1}{\sqrt{1-a^2}} = \beta \Rightarrow \tan\beta = \dfrac{1}{\sqrt{1-a^2}} \right.$

$= \sec(\cot^{-1}(\sin(\beta)))$ $\quad$ $\Rightarrow \sin\beta = \dfrac{1}{\sqrt{2-a^2}}$

$\Rightarrow \quad y = \sec\left(\cot^{-1}\left(\dfrac{1}{\sqrt{2-a^2}}\right)\right)$ $\quad$ $\left\{ \text{Let } \cot^{-1}\dfrac{1}{\sqrt{2-a^2}} = \gamma \Rightarrow \cot\gamma = \dfrac{1}{\sqrt{2-a^2}} \Rightarrow \sec\gamma = \sqrt{3-a^2} \right.$

$= \sec\gamma$

$\Rightarrow \quad y = \sqrt{3-a^2}$ ...... (ii)

from (i) and (ii), $x = y = \sqrt{3-a^2}$. **Ans.**

Do yourself - 5 :

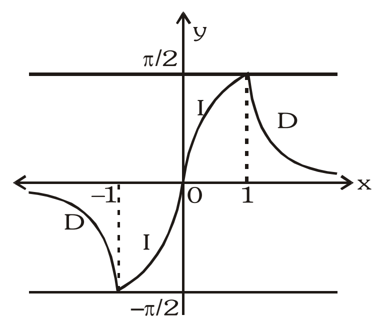
Prove the following :

(i) $\sin^{-1}\left(\dfrac{3}{5}\right) + \sin^{-1}\left(\dfrac{8}{17}\right) = \cos^{-1}\left(\dfrac{36}{85}\right)$ $\qquad$ (ii) $\tan^{-1}\left(\dfrac{3}{4}\right) + \tan^{-1}\left(\dfrac{3}{5}\right) - \tan^{-1}\left(\dfrac{8}{19}\right) = \dfrac{\pi}{4}$

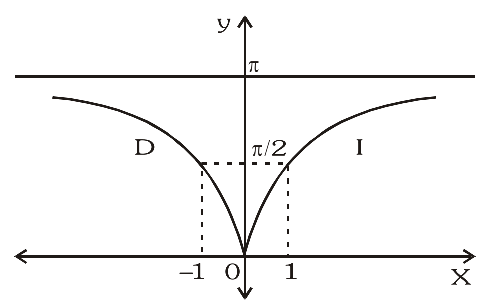
(iii) $\tan^{-1}\left(\dfrac{2}{11}\right) + \tan^{-1}\left(\dfrac{7}{24}\right) = \tan^{-1}\left(\dfrac{1}{2}\right)$

4. SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS :

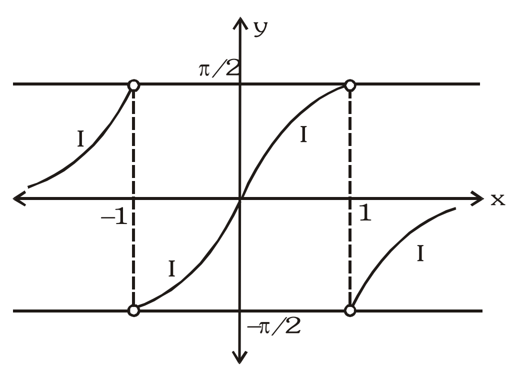
(a) $$y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$$



(b) $$y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$

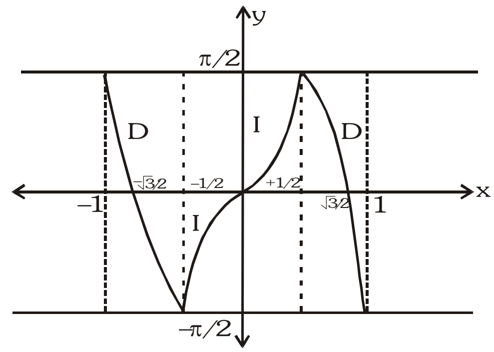


(c) $$y = f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$$



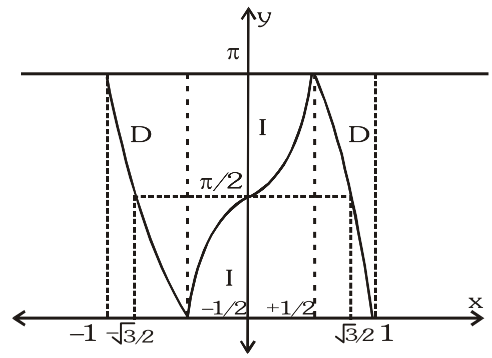
(d) $y = f(x) = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -(\pi + 3\sin^{-1} x) & \text{if } -1 \le x \le -\frac{1}{2} \\ 3\sin^{-1} x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1} x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

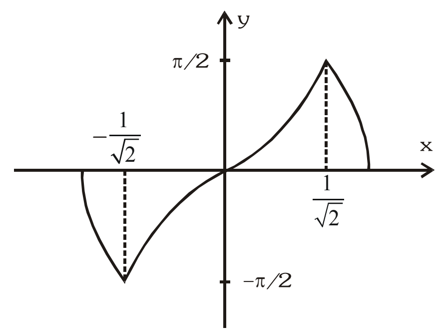


(e) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1} x - 2\pi & \text{if } -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1} x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$



(f) $$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} -\left(\pi + 2\sin^{-1} x\right) & -1 \le x \le -\frac{1}{\sqrt{2}} \\ 2\sin^{-1} x & -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1} x & \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$$



(g) $$\cos^{-1}\left(2x^2 - 1\right) = \begin{cases} 2\cos^{-1} x & 0 \le x \le 1 \\ 2\pi - 2\cos^{-1} x & -1 \le x \le 0 \end{cases}$$

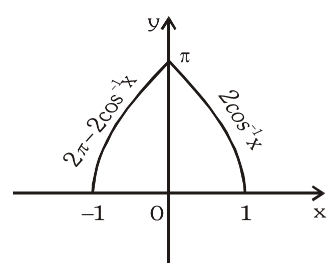


Illustration 14 **:** Evalulate : (i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$ (ii) $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$

Solution **:** (i) $$\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\} = \tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1}1\right\} \quad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } |x|<1\right]$$

$$= \tan\left\{\tan^{-1}\frac{5}{12} - \tan^{-1}1\right\} = \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\} = \tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\} = \frac{-7}{17}$$

(ii) Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$, $0 < \theta < \pi/2$

Now, $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

$$= \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sqrt{1-\frac{\sqrt{5}}{3}}}{\sqrt{1+\frac{\sqrt{5}}{3}}} = \sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}} = \sqrt{\frac{(3-\sqrt{5})^2}{(3+\sqrt{5})(3-\sqrt{5})}} = \sqrt{\frac{(3-\sqrt{5})^2}{9-5}} = \frac{3-\sqrt{5}}{2}$$

Illustration 15 : Prove that : $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Solution : We have, $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\left\{\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\} + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } -1 < x < 1\right]$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\} = \tan^{-1}\frac{31}{17}$$

Illustration 16 : Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in [0,1]$

Solution : We have, $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}\left\{\frac{1-\left(\sqrt{x}\right)^2}{1+\left(\sqrt{x}\right)^2}\right\} = \frac{1}{2}\times 2\tan^{-1}\sqrt{x} = \tan^{-1}\sqrt{x}$.

Alter : Putting $\sqrt{x} = \tan\theta$, we have $\Rightarrow \theta \in \left[0, \frac{\pi}{4}\right]$

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta) = \theta \qquad \because \left(2\theta \in \left[0, \frac{\pi}{2}\right]\right)$$

$$= \tan^{-1}\sqrt{x} = \text{LHS}$$

Illustration 17 : Prove that : (i) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$

(ii) $2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Solution : (i) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = 2\left\{2\tan^{-1}\frac{1}{5}\right\} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$

$$= 2\left\{\tan^{-1}\frac{2\times 1/5}{1-(1/5)^2}\right\} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} \qquad \left[\begin{array}{c}\because 2\tan^{-1}x \\ = \tan^{-1}\dfrac{2x}{1-x^2}, \text{if } |x|<1\end{array}\right]$$

$$= 2\tan^{-1}\frac{5}{12} - \left\{\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}\right\} = \tan^{-1}\left\{\frac{2\times 5/12}{1-(5/12)^2}\right\} - \tan^{-1}.\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\}$$

$$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{29}{6931} = \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239} = \tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\} = \tan^{-1}1 = \frac{\pi}{4}$$

(ii) $$2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8} = 2\left\{\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right\} + \sec^{-1}\frac{5\sqrt{2}}{7}$$

$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1} \qquad \left[\because \sec^{-1}x = \tan^{-1}\sqrt{x^2-1}\right]$$

$$= 2\tan^{-1}\frac{13}{39} + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left\{\frac{2\times 1/3}{1-(1/3)^2}\right\} + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, \text{ if} |x|<1\right]$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\} = \tan^{-1}1 = \frac{\pi}{4}$$

Do yourself - 6 :

Prove the following results :

(i) $2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{4}{7}\right)$ (ii) $2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

6. EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS :

Illustration 18 : The equation $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$ has

(A) no solution (B) only one solution (C) two solutions (D) three solutions

Solution : Given equation is $2\cos^{-1}x + \sin^{-1}x = \frac{11\pi}{6}$

$\Rightarrow \cos^{-1}x + (\cos^{-1}x + \sin^{-1}x) = \frac{11\pi}{6} \Rightarrow \cos^{-1}x + \frac{\pi}{2} = \frac{11\pi}{6} \Rightarrow \cos^{-1}x = 4\pi/3$

which is not possible as $\cos^{-1}x \in [0, \pi]$ **Ans.(A)**

Illustration 19 : If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = 5\pi^2 / 8$, then x is equal to-

(A) –1 (B) 0 (C) 1 (D) none of these

Solution : The given equation can be written as $(\tan^{-1} x + \cot^{-1} x)^2 - 2\tan^{-1} x \cot^{-1} x = 5\pi^2/8$

Since $\tan^{-1} x + \cot^{-1} x = \pi/2$ we have

$(\pi/2)^2 - 2\tan^{-1} x\,(\pi/2 - \tan^{-1} x) = 5\pi^2/8$

$\Rightarrow \quad 2(\tan^{-1} x)^2 - 2(\pi/2)\tan^{-1} x - 3\pi^2/8 = 0 \quad \Rightarrow \quad \tan^{-1} x = -\pi/4 \Rightarrow x = -1$ **Ans. (A)**

Illustration 20 : Solve the equation : $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$

Solution : $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$

taking tangent on both sides

$$\Rightarrow \quad \tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right)\right) = 1 \quad \Rightarrow \quad \frac{\tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)}{1 - \tan\left(\tan^{-1}\left(\frac{x-1}{x-2}\right)\right)\tan\left(\tan^{-1}\left(\frac{x+1}{x+2}\right)\right)} = 1$$

$$\Rightarrow \quad \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2}\cdot\frac{x+1}{x+2}} = 1 \quad \Rightarrow \quad \frac{(x-1)(x+2) + (x-2)(x+1)}{x^2 - 4 - (x^2 - 1)} = 1 \quad \Rightarrow \quad 2x^2 - 4 = -3 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

Now verify $x = \dfrac{1}{\sqrt{2}}$

$$= \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}} - 1}{\frac{1}{\sqrt{2}} - 2}\right) + \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}} + 2}\right) = \tan^{-1}\left(\frac{\sqrt{2} - 1}{2\sqrt{2} - 1}\right) + \tan^{-1}\left(\frac{\sqrt{2} + 1}{2\sqrt{2} + 1}\right)$$

$$= \tan^{-1}\left(\frac{(2\sqrt{2}+1)(\sqrt{2}-1) + (2\sqrt{2}-1)(\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1) - (\sqrt{2}-1)(\sqrt{2}+1)}\right) = \tan^{-1}\left(\frac{6}{6}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$x = -\dfrac{1}{\sqrt{2}}$

$$= \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}} - 1}{-\frac{1}{\sqrt{2}} - 2}\right) + \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}} + 1}{-\frac{1}{\sqrt{2}} + 2}\right) = \tan^{-1}\left(\frac{\sqrt{2} + 1}{2\sqrt{2} + 1}\right) + \tan^{-1}\left(\frac{\sqrt{2} - 1}{2\sqrt{2} - 1}\right) \quad \text{\{same as above\}}$$

$= \tan^{-1}(1) = \dfrac{\pi}{4}$

$\therefore \quad x = \pm\dfrac{1}{\sqrt{2}}$ are solutions **Ans.**

Illustration 21 : Solve the equation : $2\tan^{-1}(2x + 1) = \cos^{-1} x$.

Solution : Here, $2\tan^{-1}(2x + 1) = \cos^{-1} x$

or $\cos(2\tan^{-1}(2x + 1)) = x$ $\quad \left\{\text{We know } \cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}\right\}$

$\therefore \quad \dfrac{1-(2x+1)^2}{1+(2x+1)^2} = x \quad \Rightarrow \quad (1 - 2x - 1)(1 + 2x + 1) = x(4x^2 + 4x + 2)$

$\Rightarrow \quad -2x \,.\, 2(x + 1) = 2x(2x^2 + 2x + 1) \quad \Rightarrow \quad 2x(2x^2 + 2x + 1 + 2x + 2) = 0$

$\Rightarrow \quad 2x(2x^2 + 4x + 3) = 0$

$\Rightarrow \quad x = 0 \quad \text{or} \quad 2x^2 + 4x + 3 = 0 \quad \{\text{No solution}\}$

Verify $x = 0$

$2\tan^{-1}(1) = \cos^{-1}(1) \quad \Rightarrow \quad \dfrac{\pi}{2} = \dfrac{\pi}{2}$

$\therefore$ $x = 0$ is only the solution **Ans.**

Do yourself - 7 :

Solve the following equation for x :

(i) $\sin\left[\sin^{-1}\left(\dfrac{1}{5}\right) + \cos^{-1} x\right] = 1$ (ii) $\cos^{-1} x + \sin^{-1}\dfrac{x}{2} = \dfrac{\pi}{6}$

(iii) $\cot^{-1} x - \cot^{-1}(x+2) = \dfrac{\pi}{12}$, where $x > 0$.

7. INEQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTION :

Illustration 22 **:** Find the complete solution set of $\sin^{-1}(\sin 5) > x^2 - 4x$.

Solution :

$\sin^{-1}(\sin 5) > x^2 - 4x \quad \Rightarrow \quad \sin^{-1}[\sin(5 - 2\pi)] > x^2 - 4x$

$\Rightarrow \quad x^2 - 4x < 5 - 2\pi \quad \Rightarrow \quad x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow \quad 2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi} \quad \Rightarrow \quad x \in (2 - \sqrt{9-2\pi},\ 2 + \sqrt{9-2\pi})$

Ans.

Illustration 23 **:** Find the complete solution set of $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$, where [.] denotes the greatest integer function.

Solution :

$[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$

$\Rightarrow \quad ([\cot^{-1}x] - 3)^2 \le 0 \quad \Rightarrow \quad [\cot^{-1}x] = 3 \quad \Rightarrow \quad 3 \le \cot^{-1}x < 4 \quad \Rightarrow \quad x \in (-\infty, \cot 3]$

Illustration 24 **:** If $\cot^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{6}, n \in N$, then the maximum value of n is -

(A) 1 (B) 5 (C) 9 (D) none of these

Solution :

$\cot^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{6}$

$\Rightarrow \quad \cot\left(\cot^{-1}\left(\dfrac{n}{\pi}\right)\right) < \cot\left(\dfrac{\pi}{6}\right) \Rightarrow \dfrac{n}{\pi} < \sqrt{3}$

$\Rightarrow \quad n < \pi\sqrt{3} \quad \Rightarrow \quad n < 5.5 \text{ (approx)}$

$\Rightarrow \quad n = 5 \quad \because (n \in N)$ **Ans. (B)**

Do yourself - 8 :

(i) Solve the inequality $\tan^{-1}x > \cot^{-1}x$.

(ii) Complete solution set of inequation $(\cos^{-1}x)^2 - (\sin^{-1}x)^2 > 0$, is

(A) $\left[0, \dfrac{1}{\sqrt{2}}\right)$ (B) $\left[-1, \dfrac{1}{\sqrt{2}}\right)$ (C) $(-1, \sqrt{2})$ (D) none of these

8. SUMMATION OF SERIES :

Illustration 25 : Prove that :

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)$$

Solution : L.H.S. $\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\ldots+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$

$$=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}.\frac{1}{c_1}}\right)+\tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1}.\frac{1}{c_2}}\right)+\tan^{-1}\left(\frac{\frac{1}{c_2}-\frac{1}{c_3}}{1+\frac{1}{c_2}.\frac{1}{c_3}}\right)+\ldots+\tan^{-1}\left(\frac{\frac{1}{c_{n-1}}-\frac{1}{c_n}}{1+\frac{1}{c_{n-1}}.\frac{1}{c_n}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\left(\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}\right)+\left(\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2}\right)+\left(\tan^{-1}\frac{1}{c_2}\tan^{-1}\frac{1}{c_3}\right)+\ldots\ldots$$

$$+\left(\tan^{-1}\frac{1}{c_{n-1}}-\tan^{-1}\frac{1}{c_n}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left(\frac{x}{y}\right)=\text{R.H.S.}$$

Do yourself - 9 :

(i) Evaluate : $\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2}{1+(2r+1)(2r-1)}\right)$

Miscellaneous Illustrations :

Illustration 26 : If $\tan^{-1} y = 4\tan^{-1} x, \left(|x|<\tan\frac{\pi}{8}\right)$, find y as an algebraic function of x and hence prove that $\tan\frac{\pi}{8}$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

Solution : We have $\tan^{-1} y = 4\tan^{-1} x$

$\Rightarrow \quad \tan^{-1} y = 2\tan^{-1}\frac{2x}{1-x^2}$ (as $|x| < 1$)

$= \tan^{-1}\dfrac{\frac{4x}{1-x^2}}{1-\frac{4x^2}{(1-x^2)^2}} = \tan^{-1}\dfrac{4x(1-x^2)}{x^4-6x^2+1}$ $\left(\text{as}\left|\frac{2x}{1-x^2}\right|<1\right)$

$\Rightarrow \quad y = \dfrac{4x(1-x^2)}{x^4-6x^2+1}$

If $x = \tan\frac{\pi}{8}$

$\Rightarrow \quad \tan^{-1} y = 4\tan^{-1} x = \frac{\pi}{2} \quad \Rightarrow \quad$ y is not defined $\quad \Rightarrow \quad x^4 - 6x^2 + 1 = 0$ **Ans.**

Illustration 27 : If $A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$ and $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$, then show A > B.

Solution : We have, $A = 2\tan^{-1}\left(2\sqrt{2}-1\right) = 2\tan^{-1}(1.828)$

$\Rightarrow \quad A > 2\tan^{-1}\left(\sqrt{3}\right) \quad \Rightarrow \quad A > \frac{2\pi}{3}$ .... (i) also we have, $\sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow \sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{6}$

$\Rightarrow \quad 3\sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{2}$

also, $3\sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(3.\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right) = \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$

$\Rightarrow \quad 3\sin^{-1}(1/3) < \sin^{-1}\left(\sqrt{3}/2\right) \quad \Rightarrow \quad 3\sin^{-1}(1/3) < \pi/3$

also, $\sin^{-1}(3/5) = \sin^{-1}(0.6) < \sin^{-1}\left(\sqrt{3}/2\right) \Rightarrow \quad \sin^{-1}(3/5) < \pi/3$

Hence, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) < \frac{2\pi}{3}$ .... (ii)

From (i) and (ii), we have $A > B$.

Illustration 28 : Solve for x : If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$, where [.] denotes the greatest integer function.

Solution : We have, $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$

$\Rightarrow \quad 1 \le \sin^{-1}.\cos^{-1}.\sin^{-1}.\tan^{-1}x \le \frac{\pi}{2} \quad \Rightarrow \quad \sin 1 \le \cos^{-1}.\sin^{-1}.\tan^{-1}x \le 1$

$\Rightarrow \quad \cos\sin 1 \ge \sin^{-1}.\tan^{-1}x \ge \cos 1 \quad \Rightarrow \quad \sin\cos\sin 1 \ge \tan^{-1}x \ge \sin\cos 1$

$\Rightarrow \quad \tan\sin\cos\sin 1 \ge x \ge \tan\sin\cos 1$

Hence, $x \in [\tan\sin\cos 1, \tan\sin\cos\sin 1]$ **Ans.**

Illustration 29 : If $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$ then find the sum of all possible values of $\tan\theta$.

Solution : $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) \quad \Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left[\frac{2\left(\frac{1}{3}\tan\theta\right)}{1+\left(\frac{1}{3}\tan\theta\right)^2}\right] \quad \Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{2}{2}\tan^{-1}\left(\frac{1}{3}\tan\theta\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{1}{3}\tan\theta\right)$ .... (i)

taking tangent on both sides

$\Rightarrow \quad \tan\theta = \frac{6\tan^2\theta - \tan\theta}{3+2\tan^3\theta} \quad \Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 4\tan\theta = 0$

$\Rightarrow \quad 2\tan\theta(\tan^3\theta - 3\tan\theta + 2) = 0 \quad \Rightarrow \quad 2\tan\theta(\tan\theta - 1)^2(\tan\theta + 2) = 0$

$\Rightarrow \quad \tan\theta = 0, 1, -2$ which satisfy equation (i)

$\therefore \quad$ sum $= 0 + 1 - 2 = -1$ **Ans.**

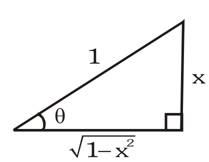
Illustration 30 : Transform $\sin^{-1}x$ in other inverse trigonometric functions, where $x \in (-1, 1) - \{0\}$

Solution : **Case-I :** $0 < x < 1$

Let $\sin^{-1}x = \theta, \quad \theta \in \left(0, \frac{\pi}{2}\right)$

Now, $\cos\theta = \sqrt{1-\sin^2\theta} \quad \Rightarrow \quad \theta = \cos^{-1}\sqrt{1-x^2}$

$\Rightarrow \quad \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$



$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$\Rightarrow \quad \theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}} \qquad \Rightarrow \quad \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \quad \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Hence, $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

$$= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right),\ 0 < x < 1$$

Case-II : $\quad -1 < x < 0$

Let $\sin^{-1}x = \theta \qquad \theta \in \left(-\frac{\pi}{2}, 0\right)$

Then $x = \sin\theta$

$\Rightarrow \quad \cos\theta = \sqrt{1-x^2} \qquad \Rightarrow \quad \cos(-\theta) = \sqrt{1-x^2}$

$\Rightarrow \quad \theta = -\cos^{-1}\sqrt{1-x^2} \qquad \Rightarrow \quad \sin^{-1}x = -\cos^{-1}\sqrt{1-x^2} = -\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

Again, $\tan\theta = \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \quad \theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}} \qquad \Rightarrow \quad \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \quad \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = -\pi + \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \qquad \left[\because\ \tan^{-1}x = -\pi + \cot^{-1}\left(\frac{1}{x}\right),\ x < 0\right]$

Hence, $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$

$$= -\sec^{-1}\frac{1}{\sqrt{1-x^2}} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = -\pi + \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right),\ -1 < x < 0$$

ANSWERS FOR DO YOURSELF

1 :	(i)	C	(ii)	1526								
2 :	(i)	$\frac{15}{8}$	(ii)	$\frac{1}{\sqrt{10}}$	(iii)	$\frac{4}{5}$	(iv)	1	(v)	$\frac{\sqrt{3}}{2}$	(vi)	$\frac{4}{5}$
3 :	(i)	$\frac{\pi}{6}$	(ii)	$\frac{\pi}{6}$	(iii)	$\pi - 2$	(iv)	$\frac{\pi}{6}$				
	(v)	$-\frac{\pi}{3}$	(vi)	$\frac{-\pi}{4}$	(vii)	$\frac{2\pi}{3}$						
4 :	(ii)	$\begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \end{cases}$										
7 :	(i)	$\frac{1}{5}$	(ii)	1	(iii)	$\sqrt{3}$						
8 :	(i)	$(1, \infty)$	(ii)	B								
9 :	(i)	$\pi/4$										

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. The value of $\sin^{-1}(-\sqrt{3}/2)$ is -
(A) $-\pi/3$ (B) $-2\pi/3$ (C) $4\pi/3$ (D) $5\pi/3$

2. $\cos\left(2\tan^{-1}\left(\frac{1}{7}\right)\right)$ equals -
(A) $\sin(4\cot^{-1}3)$ (B) $\sin(3\cot^{-1}4)$ (C) $\cos(3\cot^{-1}4)$ (D) $\cos(4\cot^{-1}4)$

3. The value of $\sec\left[\sin^{-1}\left(-\sin\frac{50\pi}{9}\right)+\cos^{-1}\cos\left(-\frac{31\pi}{9}\right)\right]$ is equal to -
(A) $\sec\frac{10\pi}{9}$ (B) $\sec\frac{\pi}{9}$ (C) 1 (D) -1

4. $\cos\left(\cos^{-1}\cos\left(\frac{8\pi}{7}\right)+\tan^{-1}\tan\left(\frac{8\pi}{7}\right)\right)$ has the value equal to -
(A) 1 (B) -1 (C) $\cos\frac{\pi}{7}$ (D) 0

5. $(\sin^{-1}x)^2+(\sin^{-1}y)^2+2(\sin^{-1}x)(\sin^{-1}y)=\pi^2$, then x^2+y^2 is equal to -
(A) 1 (B) 3/2 (C) 2 (D) 1/2

6. $\cot^{-1}[(\cos\alpha)^{1/2}]-\tan^{-1}[(\cos\alpha)^{1/2}]=x$, then $\sin x$ =
(A) $\tan^2\left(\frac{\alpha}{2}\right)$ (B) $\cot^2\left(\frac{\alpha}{2}\right)$ (C) $\tan\alpha$ (D) $\cot\left(\frac{\alpha}{2}\right)$

7. $\tan(\cos^{-1}x)$ is equal to
(A) $\frac{x}{1+x^2}$ (B) $\frac{\sqrt{1+x^2}}{x}$ (C) $\frac{\sqrt{1-x^2}}{x}$ (D) $\sqrt{1-2x}$

8. If $x = 2\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\tan^{-1}\left(\sqrt{3}\right)$ and $y=\cos\left(\frac{1}{2}\sin^{-1}\left(\sin\frac{x}{2}\right)\right)$ then which of the following statements holds good ?
(A) $y=\cos\frac{3\pi}{16}$ (B) $y=\cos\frac{5\pi}{16}$ (C) $x=4\cos^{-1}y$ (D) none of these

9. If $x=\tan^{-1}1-\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\frac{1}{2}$; $y=\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$ then -
(A) $x=\pi y$ (B) $y=\pi x$ (C) $\tan x=-(4/3)y$ (D) $\tan x=(4/3)y$

10. $\tan^{-1}2+\tan^{-1}3=\operatorname{cosec}^{-1}x$, then x is equal to -
(A) 4 (B) $\sqrt{2}$ (C) $-\sqrt{2}$ (D) none of these

11. The number k is such that $\tan\{\arctan(2)+\arctan(20k)\}=k$. The sum of all possible values of k is -
(A) $-\frac{19}{40}$ (B) $-\frac{21}{40}$ (C) 0 (D) $\frac{1}{5}$

12. If $\sin^{-1}x+\cot^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}$, then x is -
(A) 0 (B) $\frac{1}{\sqrt{5}}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{3}}{2}$

13. If $\tan(\cos^{-1}x) = \sin(\cot^{-1} 1/2)$ then x is equal to -

(A) $1/\sqrt{5}$ (B) $2/\sqrt{5}$ (C) $3/\sqrt{5}$ (D) $\sqrt{5}/3$

14. $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$ is true if -

(A) $x \in [0,1]$ (B) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (D) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

15. Domain of the explicit form of the function y represented implicitly by the equation $(1+x)\cos y - x^2 = 0$ is -

(A) $(-1,1]$ (B) $\left(-1, \frac{1-\sqrt{5}}{2}\right]$ (C) $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$ (D) $\left[0, \frac{1+\sqrt{5}}{2}\right]$

16. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to -

(A) $-4\sin^2\alpha$ (B) $4\sin^2\alpha$ (C) 4 (D) $2\sin 2\alpha$

17. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then -

(A) $x^2+y^2+z^2+xyz = 0$ (B) $x^2+y^2+z^2+xyz = 1$ (C) $x^2+y^2+z^2+2xyz = 0$ (D) $x^2+y^2+z^2+2xyz = 1$

18. If $\tan^{-1}\frac{x}{\pi} < \frac{\pi}{3}, x \in N$, then the maximum value of x is -

(A) 2 (B) 5 (C) 7 (D) none of these

19. The solution of the inequality $(\tan^{-1}x)^2 - 3\tan^{-1}x + 2 \geq 0$ is -

(A) $(-\infty, \tan 1] \cup [\tan 2, \infty)$ (B) $(-\infty, \tan 1]$ (C) $(-\infty, -\tan 1] \cup [\tan 2, \infty)$ (D) $[\tan 2, \infty)$

20. The set of values of x, satisfying the equation $\tan^2(\sin^{-1}x) > 1$ is -

(A) $[-1,1]$ (B) $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ (C) $(-1,1) - \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ (D) $[-1,1] - \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

<u>SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)</u>

21. If numerical value of $\tan\left\{\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right\}$ is $\frac{a}{b}$, then -

(A) $a + b = 23$ (B) $a - b = 11$ (C) $3b = a + 1$ (D) $2a = 3b$

22. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left(\cos\left(-\frac{14\pi}{5}\right)\right)\right]$ is/are -

(A) $\cos\left(-\frac{7\pi}{5}\right)$ (B) $\sin\left(\frac{\pi}{10}\right)$ (C) $\cos\left(\frac{2\pi}{5}\right)$ (D) $-\cos\left(\frac{3\pi}{5}\right)$

23. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$ equals to

(A) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (C) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (D) $\tan^{-1}\left(\frac{1}{2}\right)$

24. $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1}x$, then roots of the equation are -

(A) 0 (B) 1 (C) –1 (D) –2

CHECK YOUR GRASP — **ANSWER KEY** — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	A	A	D	B	C	A	C	A	C	D	A	B	D	B	C
Que.	16	17	18	19	20	21	22	23	24						
Ans.	B	D	B	B	C	A,B,C	B,C,D	A,D	A,B,C						

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. $\cos^{-1}x = \tan^{-1}x$ then -

(A) $x^2 = \left(\frac{\sqrt{5}-1}{2}\right)$ (B) $x^2 = \left(\frac{\sqrt{5}+1}{2}\right)$

(C) $\sin(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$ (D) $\tan(\cos^{-1}x) = \left(\frac{\sqrt{5}-1}{2}\right)$

2. The value of $\sin\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right)+\cos\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right)$ is/are equal to -

(A) 1 (B) $\frac{3\sqrt{2}}{\sqrt{10}}$

(C) $\sqrt{2}\sin\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)+\cot^{-1}(1)\right)$ (D) $\sqrt{2}\sin\left(\pi-\tan^{-1}(1)-\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$

3. The value of $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$ for $0 < A < (\pi/4)$ is -

(A) $4\tan^{-1}(1)$ (B) $2\tan^{-1}(2)$ (C) 0 (D) none

4. For the equation $2x = \tan(2\tan^{-1}a) + 2\tan(\tan^{-1}a+\tan^{-1}a^3)$, which of the following is/are invalid ?

(A) $a^2x + 2a = x$ (B) $a^2 + 2ax + 1 = 0$ (C) $a \neq 0$ (D) $a \neq -1, 1$

5. The value of $\left[\tan\left\{\frac{\pi}{4}+\frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right\}\right]^{-1}$, where ($0 < a < b$), is -

(A) $\frac{b}{2a}$ (B) $\frac{a}{2b}$ (C) $\frac{\sqrt{b^2-a^2}}{2b}$ (D) $\frac{\sqrt{b^2-a^2}}{2a}$

6. Identify the pair(s) of functions which are identical -

(A) $y = \tan(\cos^{-1}x)$; $y = \frac{\sqrt{1-x^2}}{x}$ (B) $y = \tan(\cot^{-1}x)$; $y = \frac{1}{x}$

(C) $y = \sin(\text{arc}\tan x)$; $y = \frac{x}{\sqrt{1+x^2}}$ (D) $y = \cos(\text{arc}\tan x)$; $y = \sin(\text{arc}\cot x)$

7. Which of the following, satisfy the equation $2\cos^{-1}x = \cot^{-1}\left(\frac{2x^2-1}{\sqrt{4x^2-4x^4}}\right)$

(A) $(-1, 0)$ (B) $(0, 1)$ (C) $\left(-\frac{1}{2},\frac{1}{\sqrt{2}}\right)$ (D) $[-1, 1]$

8. The solution set of the equation $\sin^{-1}\sqrt{1-x^2}+\cos^{-1}x = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) - \sin^{-1}x$ is -

(A) $[-1, 1]-\{0\}$ (B) $(0,1] \cup \{-1\}$ (C) $[-1,0) \cup \{1\}$ (D) $[-1,1]$

9. If $0 < x < 1$, then $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ is equal to -

(A) $\frac{1}{2}\cos^{-1}x$ (B) $\cos^{-1}\sqrt{\frac{1+x}{2}}$ (C) $\sin^{-1}\sqrt{\frac{1-x}{2}}$ (D) $\frac{1}{2}\tan^{-1}\sqrt{\frac{1+x}{1-x}}$

10. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is - [JEE 99]

(A) zero (B) one (C) two (D) infinite

11. If $[\sin^{-1}x] + [\cos^{-1}x] = 0$, where 'x' is a non negative real number and [.] denotes the greatest integer function, then complete set of values of x is -

(A) (cos1, 1) (B) (–1, cos1) (C) (sin1, 1) (D) (cos1, sin1)

12. Value of k for which the point $(\alpha, \sin^{-1}\alpha)(\alpha > 0)$ lies inside the triangle formed by x + y = k with co-ordinate axes is -

(A) $\left(1+\frac{\pi}{2},\infty\right)$ (B) $\left(-\left(1+\frac{\pi}{2}\right), \left(1+\frac{\pi}{2}\right)\right)$ (C) $\left(-\infty\ ,\ 1+\frac{\pi}{2}\right)$ (D) (–1–sin1, 1+sin1)

13. Solution set of the inequality $\sin^{-1}\left(\sin\frac{2x^2+3}{x^2+1}\right) \le \pi - \frac{5}{2}$ is -

(A) $(-\infty, 1) \cup (1, \infty)$ (B) $[-1, 1]$ (C) $(-1, 1)$ (D) $(-\infty, -1] \cup [1, \infty)$

14. Consider two geometric progressions $a_1, a_2, a_3.......a_n$ & $b_1, b_2, b_3,.....b_n$ with $a_r = \frac{1}{b_r} = 2^{r-1}$ and another sequence $t_1, t_2, t_3.......t_n$ such that $t_r = \cot^{-1}(2a_r + b_r)$ then $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n} t_r$ is -

(A) 0 (B) $\pi/4$ (C) $\tan^{-1}2$ (D) $\pi/2$

15. The sum of the infinite terms of the series -

$\cot^{-1}\left(1^2+\frac{3}{4}\right) + \cot^{-1}\left(2^2+\frac{3}{4}\right) + \cot^{-1}\left(3^2+\frac{3}{4}\right) + ...$ is equal to -

(A) $\tan^{-1}(1)$ (B) $\tan^{-1}(2)$ (C) $\tan^{-1}(3)$ (D) $\frac{3\pi}{4} - \tan^{-1}3$

BRAIN TEASERS — **ANSWER KEY** — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A,C	B,C,D	A	B,C	C	A,B,C,D	B	C	A,B,C	C
Que.	11	12	13	14	15					
Ans.	D	A	B	B	B,D					

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

FILL IN THE BLANKS

1. $\tan\left[\cos^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$ = ...

2. $\cos\ (\tan^{-1}2)$ = ...

3. $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$ = ...

4. $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$ = ...

5. $\sin^{-1}\frac{3}{\sqrt{73}}+\cos^{-1}\frac{11}{\sqrt{146}}+\cot^{-1}\sqrt{3}$ = ...

6. $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)-\cos^{-1}\left(\frac{1}{\sqrt{10}}\right)-\cot^{-1}\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$ = ...

7. $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right]$ = ...

8. $-\left(\cos^{-1}\frac{1}{\sqrt{3}}+\cos^{-1}\frac{1}{\sqrt{6}}\right)-\cos^{-1}\left(\frac{\sqrt{10}\ -1}{3\sqrt{2}}\right)+4\cot^{-1}1$ = ...

9. $\tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right]+\tan^{-1}\left[\frac{\tan\alpha}{4}\right]$, where $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$ = ...

10. The number of roots of the equation $\sqrt{\sin x}=\cos^{-1}(\cos x)$ is ....

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1.

	Column-I		Column-II
(A)	$\sin^{-1}\left(\sin\frac{33\pi}{7}\right)$	(p)	$-2\pi/7$
(B)	$\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$	(q)	$2\pi/7$
(C)	$\tan^{-1}\left(\tan\left(\frac{-33\pi}{7}\right)\right)$	(r)	$3\pi/7$
(D)	$\cot^{-1}\left(\cot\left(\frac{-46\pi}{7}\right)\right)$	(s)	$4\pi/7$

2.

	Column-I		Column-II
(A)	$\sin(\tan^{-1}x)$	(p)	x
(B)	$\cos(\tan^{-1}x)$	(q)	$\frac{x}{\sqrt{x^2+1}}$
(C)	$\sin(\cot^{-1}(\tan(\cos^{-1}x))).\ x\in(0,1]$	(r)	$\frac{1}{\sqrt{x^2+1}}$
(D)	$\sin(\mathrm{cosec}^{-1}(\cot(\tan^{-1}x)))$; $x\in(0,1]$	(s)	$\sqrt{1-x^2}$

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

3. $x \geq 0$, $y \geq 0$, $z \geq 0$ and $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = k$, the possible value(s) of k, if

	Column-I		Column-II
(A)	$xy + yz + zx = 1$, then	(p)	$k = \frac{\pi}{2}$
(B)	$x + y + z = xyz$, then	(q)	$k = \pi$
(C)	$x^2 + y^2 + z^2 = 1$ and $x + y + z = \sqrt{3}$, then	(r)	$k = 0$
(D)	$x = y = z$ and $xyz \geq 3\sqrt{3}$, then	(s)	$k = \frac{7\pi}{6}$

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.
(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.
(C) Statement-I is true, Statement-II is false.
(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : Range of $\cos\left(\sec^{-1}\frac{1}{x}+\operatorname{cosec}^{-1}\frac{1}{x}+\tan^{-1}x\right)$ is $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

Because

Statement-II : Range of $\sin^{-1}x + \tan^{-1}x + \cos^{-1}x$ is $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$.

(A) A (B) B (C) C (D) D

2. **Statement-I** : If r, s & t be the roots of the equation : $x(x - 2)(3x - 7) = 2$, then $\tan^{-1}r + \tan^{-1}s + \tan^{-1}t = 3\pi/4$.

Because

Statement-II : The roots of the equation $x(x - 2)(3x - 7) = 2$ are real & negative.

(A) A (B) B (C) C (D) D

3. **Statement-I** : If $\sum_{i=1}^{2n}\sin^{-1}x_i = n\pi,\ n \in N$. Then $\sum_{i=1}^{n}x_i = \sum_{i=1}^{n}x_i^2 = \sum_{i=1}^{n}x_i^3$

Because

Statement-II : $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}, \forall\ x \in [-1, 1]$.

(A) A (B) B (C) C (D) D

4. Let $f : R \rightarrow [0, \pi/2)$ defined by $f(x) = \tan^{-1}(x^2 + x + a)$, then

Statement-I : The set of values of a for which $f(x)$ is onto is $\left[\frac{1}{4}, \infty\right)$.

Because

Statement-II : Minimum value of $x^2 + x + a$ is $a - \frac{1}{4}$.

(A) A (B) B (C) C (D) D

5. **Statement-I** : $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{9}{5}\right) = \pi - \frac{9}{5}$.

Because

Statement-II : $\operatorname{cosec}^{-1}(\operatorname{cosec}x) = \pi - x$; $\forall\ x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

Consider the two equations in x ; (i) $\sin\left(\frac{\cos^{-1}x}{y}\right)=1$ (ii) $\cos\left(\frac{\sin^{-1}x}{y}\right)=0$

The sets $X_1, X_2 \subseteq [-1, 1]$; $Y_1, Y_2 \subseteq I-\{0\}$ are such that

X_1 : the solution set of equation (i)

X_2 : the solution set of equation (ii)

Y_1 : the set of all integral values of y for which equation (i) possess a solution

Y_2 : the set of all intergral values of y for which equation (ii) possess a solution

Let : C_1 be the correspondence : $X_1 \to Y_1$ such that x C_1y for $x \in X_1$, $y \in Y_1$ & (x, y) satisfy (i).

C_2 be the correspondence : $X_2 \to Y_2$ such that x C_2y for $x \in X_2$, $y \in Y_2$ & (x, y) satisfy (ii).

On the basis of above information, answer the following questions :

1. The number of ordered pair (x, y) satisfying correspondence C_1 is

(A) 1 (B) 2 (C) 3 (D) 4

2. The number of ordered pair (x, y) satisfying correspondence C_2 is

(A) 1 (B) 2 (C) 3 (D) 4

3. $C_1 : X_1 \to Y_1$ is a function which is -

(A) one-one (B) many-one (C) onto (D) into

Comprehension # 2

Let $h_1(x) = \sin^{-1}(3x - 4x^3)$; $h_2(x) = \cos^{-1}(4x^3 - 3x)$ & $f(x) = h_1(x) + h_2(x)$

when $x \in [-1, \frac{-1}{2}]$; let $f(x) = a\cos^{-1}x + b\pi$; $a, b \in Q$

$h_1(x) = p\sin^{-1}x + q\pi$; $p, q \in Q$

$h_2(x) = r\cos^{-1}x + s\pi$; $r, s \in Q$

Let C_1 be the circle with centre (p, q) & radius 1 & C_2 be the circle with centre (r, s) & radius 1.

On the basis of above information, answer the following questions :

1. $p + r + 2q - s =$

(A) 0 (B) 1 (C) 2 (D) 4

2. If $b.\log_{|s|}|p+q| = k.a$, then value of k is -

(A) $\frac{9}{2}$ (B) 6 (C) $\frac{-3}{2}$ (D) none of these

3. Radical axis of circle C_1 & C_2 is -

(A) $12x - 2y - 3 = 0$ (B) $12x + 2y - 3 = 0$ (C) $-12x + 2y - 3 = 0$ (D) none of these

MISCELLANEOUS TYPE QUESTION **ANSWER KEY** EXERCISE -3

- ***Fill in the Blanks***

1. $\frac{1}{\sqrt{3}}$ **2.** $\frac{1}{\sqrt{5}}$ **3.** $\frac{17}{6}$ **4.** -1 **5.** $\frac{5\pi}{12}$ **6.** $-\pi$ **7.** $\frac{1}{2}$ **8.** 0

9. α **10.** infinite many solutions

- ***Match the Column***

1. (A) $\to$ (q), (B) $\to$ (s), (C) $\to$ (q), (D) $\to$ (r) **2.** (A) $\to$ (q), (B) $\to$ (r), (C) $\to$ (p), (D) $\to$ (p)

3. (A) $\to$ (p), (B) $\to$ (q,r), (C) $\to$ (p), (D) $\to$ (q,s)

- ***Assertion & Reason***

1. D **2.** C **3.** A **4.** D **5.** A

- ***Comprehension Based Questions***

Comprehension # 1 : **1.** B **2.** D **3.** A,C Comprehension # 2 : **1.** A **2.** C **3.** A

EXERCISE - 04 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

1. Find the domain of definition the following functions.

(Read the symbols [*] and { * } as greatest integers and fractional part functions respectively)

(a) $f(x) = \cos^{-1}\dfrac{2}{2+\sin x}$

(b) $f(x) = \dfrac{1}{x} + 2^{\arcsin x} + \dfrac{1}{\sqrt{x-2}}$

(c) $e^{\cos^{-1}x} + \cot^{-1}\left[\dfrac{x}{2}-1\right] + \dfrac{1}{2}\ \ln\{x\}$

(d) $f(x) = \sin^{-1}\left(\dfrac{x-3}{2}\right) - \log_{10}(4-x)$

(e) $f(x) = \dfrac{\sqrt{1-\sin x}}{\log_5(1-4x^2)} + \cos^{-1}(1-\{x\})$

(f) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\dfrac{3-2x}{5}\right) + \log_6(2|x|-3) + \sin^{-1}(\log_2 x)$

2. Find the domain and range of the following functions.

(Read the symbols [*] and { * } as greatest integers and fractional part function respectively)

(a) $y = \cot^{-1}(2x - x^2)$

(b) $f(x) = \sec^{-1}(\log_3 \tan x + \log_{\tan x} 3)$

(c) $f(x) = 2^{\cos^{-1}\left(\sin\left(x+\frac{\pi}{3}\right)\right)} + \left[\dfrac{\sqrt{1-2\cos x}}{2}\right]$

(d) $f(x) = \tan^{-1}\left(\log_{\frac{4}{5}}(5x^2-8x+4)\right)$

3. Draw the graph of the following functions :

(a) $f(x) = \sin^{-1}(x+2)$

(b) $g(x) = [\cos^{-1}x]$, where [ ] denotes greatest integer function.

(c) $h(x) = -|\tan^{-1}(3x)|$

4. Express $f(x) = \arccos x + \arccos\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right)$ in simplest form and hence find the values of

(a) $f\left(\dfrac{2}{3}\right)$

(b) $f\left(\dfrac{1}{3}\right)$

5. If $\cos^{-1}\dfrac{x}{a} + \cos^{-1}\dfrac{y}{b} = \alpha$ then prove that $\dfrac{x^2}{a^2} - \dfrac{2xy}{ab}\cos\alpha + \dfrac{y^2}{b^2} = \sin^2\alpha$.

6. Prove that : $\sin^{-1}\dfrac{3}{5} + \sin^{-1}\dfrac{8}{17} = \sin^{-1}\dfrac{77}{85}$

7. Prove that : $\cos^{-1}x = 2\sin^{-1}\sqrt{\dfrac{1-x}{2}} = 2\cos^{-1}\sqrt{\dfrac{1+x}{2}}$

8. Prove that : $\tan^{-1}\dfrac{2}{3} = \dfrac{1}{2}\tan^{-1}\dfrac{12}{5}$

9. Prove that : $3\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{1}{20} = \dfrac{\pi}{4} - \tan^{-1}\dfrac{1}{1985}$

10. If $\sin^2x + \sin^2y < 1$ for all x, y $\in$ R then prove that $\sin^{-1}(\tan x \,.\, \tan y) \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

11. Prove that : $\cot^{-1}\left(\dfrac{1+ab}{a-b}\right) + \cot^{-1}\left(\dfrac{1+bc}{b-c}\right) + \cot^{-1}\left(\dfrac{1+ca}{c-a}\right) = \pi$, $(a > b > c > 0)$

12. Let $\cos^{-1}x + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$. If x satisfies the cubic $ax^3 + bx^2 + cx - 1 = 0$, then find the value of a + b + c.

13. If $\alpha = 2\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ & $\beta = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$ then prove that $\alpha + \beta = \pi$. What is the value of $\alpha + \beta$ will be if $x > 1$?

14. Solve the following equations :

(a) $\sin^{-1} x + \sin^{-1} 2x = \dfrac{\pi}{3}$

(b) $\tan^{-1} \dfrac{1}{1+2x} + \tan^{-1} \dfrac{1}{1+4x} = \tan^{-1} \dfrac{2}{x^2}$

(c) $\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$

(d) $\sin^{-1} x = \cos^{-1} x + \sin^{-1}(3x-2)$

(e) $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$

(f) $2\tan^{-1} x = \cos^{-1} \dfrac{1-a^2}{1+a^2} - \cos^{-1} \dfrac{1-b^2}{1+b^2}$ $\quad a > 0, \ b > 0$

(g) $\cos^{-1} \dfrac{x^2-1}{x^2+1} + \tan^{-1} \dfrac{2x}{x^2-1} = \dfrac{2\pi}{3}$

15. Find the sum of the series :

(a) $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{2}{9} + \ldots\ldots + \tan^{-1}\dfrac{2^{n-1}}{1+2^{2n-1}} + \ldots\ldots\infty$

(b) $\cot^{-1} 7 + \cot^{-1} 13 + \cot^{-1} 21 + \cot^{-1} 31 + \ldots\ldots$ to n terms.

(c) $\tan^{-1}\dfrac{1}{x^2+x+1} + \tan^{-1}\dfrac{1}{x^2+3x+3} + \tan^{-1}\dfrac{1}{x^2+5x+7} + \tan^{-1}\dfrac{1}{x^2+7x+13} + \ldots\ldots$ to n terms.

CONCEPTUAL SUBJECTIVE EXERCISE **ANSWER KEY** EXERCISE-4(A)

1. (a) $[2n\pi, (2n+1)\pi]$; $n \in I$ (b) ϕ (not defined for any real x) (c) $(-1, 1) - \{0\}$ (d) $1 \le x < 4$

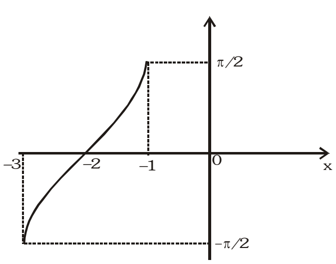
(e) $x \in (-1/2, 1/2)$, $x \ne 0$ (f) $\left(\dfrac{3}{2}, 2\right]$

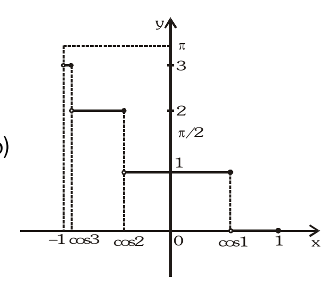
2. (a) $D : x \in R \quad R : [\pi/4, \pi)$

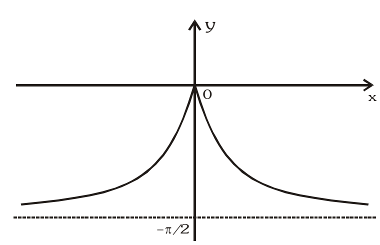
(b) $D : x \in \left(2n\pi, 2n\pi + \dfrac{\pi}{2}\right) \cup \left((2n+1)\pi, \left(2n\pi + \dfrac{3\pi}{2}\right)\right) - \left\{x \mid x = 2n\pi + \dfrac{\pi}{4} \text{ or } 2n\pi + \dfrac{5\pi}{4}\right\} n \in I$; $R : \left[\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right] - \left[\dfrac{\pi}{2}\right]$

or $n\pi < x < \pi/2 + n\pi$; $x \ne \pi/4 + n\pi$

(c) $\left[2^{\pi/6}, 2^{\pi}\right]$ (d) $D : x \in R$; $R : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{4}\right]$

3. (a)


(b)


(c)


4. (a) $\dfrac{\pi}{3}$ (b) $2\cos^{-1}\left(\dfrac{1}{3}\right) - \dfrac{\pi}{3}$ **12.** 26 **13.** $-\pi$

14. (a) $x = \dfrac{1}{2}\sqrt{\dfrac{3}{7}}$ (b) $x = 3$ (c) $x = 0, \dfrac{1}{2}, -\dfrac{1}{2}$ (d) $x = 1, \dfrac{1}{2}$ (e) $x = 0, \dfrac{1}{2}$ (f) $x = \dfrac{a-b}{1+ab}$

(g) $x = 2 - \sqrt{3}$ या $\sqrt{3}$

15. (a) $\dfrac{\pi}{4}$ (b) arc $\cot\left[\dfrac{2n+5}{n}\right]$ (c) arc tan(x + n) − arc tanx

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. Find the domain of definition the following functions.

(a) $f(x) = \log_{10}(1 - \log_7(x^2 - 5x + 13)) + \cos^{-1}\left(\dfrac{3}{2 + \sin\frac{9\pi x}{2}}\right)$

(b) $f(x) = \sqrt{\sin(\cos x)} + \ell n(-2\cos^2 x + 3\cos x + 1) + e^{\cos^{-1}\left(\frac{2\sin x + 1}{2\sqrt{2}\sin x}\right)}$

2. Prove that :

(a) $\sin^{-1}\left[\cos(\sin^{-1}x)\right] + \cos^{-1}\left[\sin(\cos^{-1}x)\right] = \dfrac{\pi}{2}, |x| \le 1$

(b) $\tan^{-1}\left(\dfrac{m}{n}\right) + \tan^{-1}\left(\dfrac{n-m}{n+m}\right) = \begin{bmatrix} \dfrac{\pi}{4} & \dfrac{m}{n} > -1 \\ \dfrac{-3\pi}{4} & \dfrac{m}{n} < -1 \end{bmatrix}$

3. Prove that : $\sin^{-1}\dfrac{1}{\sqrt{2}} + \sin^{-1}\dfrac{\sqrt{2}-1}{\sqrt{6}} + ...+ \sin^{-1}\dfrac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} + ...\infty = \dfrac{\pi}{2}$

4. If arc sin x + arc siny + arc sinz = π then prove that : (x , y , z > 0)

(a) $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

(b) $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2\left(x^2y^2 + y^2z^2 + z^2x^2\right)$

5. Find the integral values of K for which the system of equations ;

$\begin{bmatrix} \text{arc}\cos x + (\text{arc}\sin y)^2 = \dfrac{K\pi^2}{4} \\ (\text{arc}\sin y)^2\ (\text{arc}\cos x) = \dfrac{\pi^4}{16} \end{bmatrix}$ possesses solutions & find those solutions.

6. Express $\dfrac{\beta^3}{2}\text{cosec}^2\left[\dfrac{1}{2}\tan^{-1}\dfrac{\beta}{\alpha}\right] + \dfrac{\alpha^3}{2}\sec^2\left[\dfrac{1}{2}\tan^{-1}\dfrac{\alpha}{\beta}\right]$ as an integral polynomial in α & β.

7. Solve the following inequalities :

(a) arc $\cot^2 x$ – 5 arc cot x + 6 > 0

(b) arc sin x > arc cos x

(c) 4 arc $\tan^2 x$ – 8 arc tan x + 3 < 0 & 4 arc cot x – arc $\cot^2 x$ – 3 ≥ 0

8. Find all the positive integral solutions of, $\tan^{-1}x + \cos^{-1}\dfrac{y}{\sqrt{1+y^2}} = \sin^{-1}\dfrac{3}{\sqrt{10}}$.

9. Let $f(x) = \cot^{-1}(x^2 + 4x + \alpha^2 - \alpha)$ be a function defined $R \to \left(0, \dfrac{\pi}{2}\right]$ then find the complete set of real values of α for which f(x) is onto.

10. Find all values of k for which there is a triangle whose angles have measure $\tan^{-1}\left(\dfrac{1}{2}\right), \tan^{-1}\left(\dfrac{1}{2}+k\right)$ and $\tan^{-1}\left(\dfrac{1}{2}+2k\right)$.

11. Find the range of the function $f(x) = (\sin^{-1}x)^3 + (\cos^{-1}x)^3$.

12. Find the number of roots of the following equations :

(a) $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$

(b) $\sin\left(\sin^{-1}(\log_{\frac{1}{2}} x)\right) + 2\left|\cos\left(\sin^{-1}\left(\dfrac{x}{2}-1\right)\right)\right| = 0$

(c) $|y| = \cos x$ and $y = \cot^{-1}(\cot x)$ in $\left(\dfrac{-3\pi}{2}, \dfrac{5\pi}{2}\right)$

BRAIN STORMING SUBJECTIVE EXERCISE **ANSWER KEY** EXERCISE-4(B)

1. (a) $\dfrac{21}{9}, \dfrac{25}{9}$ (b) $2n\pi + \dfrac{\pi}{6}$; $n \in I$ **5.** K = 2 ; $\cos\dfrac{\pi^2}{4}$, 1 & $\cos\dfrac{\pi^2}{4}$, –1 **6.** $(\alpha^2 + \beta^2)(\alpha + \beta)$

7. (a) $(\cot 2, \infty)\cup(-\infty, \cot 3)$; (b) $\left(\dfrac{\sqrt{2}}{2}, 1\right]$; (c) $\left(\tan\dfrac{1}{2}, \cot 1\right]$ **8.** x = 1; y = 2 & x = 2; y = 7 **9.** $\dfrac{1 \pm \sqrt{17}}{2}$

10. $k = \dfrac{11}{4}$ **11.** $\left[\dfrac{\pi^3}{32}, \dfrac{7\pi^3}{8}\right]$ **12.** (a) Infinite ; (b) zero ; (c) 2

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. The value of $\cos^{-1}(-1) - \sin^{-1}(1)$ is- [AIEEE-2002]

(1) π (2) $\frac{\pi}{2}$ (3) $\frac{3\pi}{2}$ (4) $-\frac{3\pi}{2}$

2. The trigonometric equation $\sin^{-1} x = 2\sin^{-1}a$, has a solution for- [AIEEE-2003]

(1) $|a| \le \frac{1}{\sqrt{2}}$ (2) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$ (3) all real values of a (4) $|a| < \frac{1}{2}$

3. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to - [AIEEE-2005]

(1) $2\sin 2\alpha$ (2) 4 (3) $4\sin^2\alpha$ (4) $-4\sin^2\alpha$

4. If $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$ then a value of x is- [AIEEE-2007]

(1) 1 (2) 3 (3) 4 (4) 5

5. The value of $\cot\left(\text{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is equal to- [AIEEE-2008]

(1) $\frac{6}{17}$ (2) $\frac{3}{17}$ (3) $\frac{4}{17}$ (4) $\frac{5}{17}$

6. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then [JEE (Main)-2013]

(1) $x = y = z$ (2) $2x = 3y = 6z$ (3) $6x = 3y = 2z$ (4) $6x = 4y = 3z$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

Que.	1	2	3	4	5	6
Ans	2	1	3	2	1	1

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. Prove that $\cos\tan^{-1}\sin\cot^{-1}x = \sqrt{\dfrac{x^2+1}{x^2+2}}$ [JEE 2002 (Mains), 5]

2. Domain of $f(x) = \sqrt{\sin^{-1}(2x)+\dfrac{\pi}{6}}$ is -

(A) $\left(-\dfrac{1}{2},\dfrac{1}{2}\right]$ (B) $\left[-\dfrac{1}{4},\dfrac{3}{4}\right)$ (C) $\left[-\dfrac{1}{4},\dfrac{1}{4}\right]$ (D) $\left[-\dfrac{1}{4},\dfrac{1}{2}\right]$

[JEE 2003 (screening), 3]

3. If $\sin(\cot^{-1}(x+1)) = \cos(\tan^{-1}x)$, then x = [JEE 2004 (screening)]

(A) $-\dfrac{1}{2}$ (B) 0 (C) $\dfrac{1}{2}$ (D) 1

4. $\displaystyle\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then find the value of tan(t). [JEE 2006, 1½]

5. Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \dfrac{\pi}{2}$ [JEE 2007, 6]

Match the statements in column-I with statements in column-II and indicate your answer by darkening the appropriate bubbles in the 4 4 matrixgiven in the ORS.

	Column-I		Column-II
(A)	If a = 1 and b = 0, then (x, y)	(p)	lies on the circle $x^2 + y^2 = 1$
(B)	If a = 1 and b = 1, they (x, y)	(q)	lies on $(x^2 - 1)(y^2 - 1) = 0$
(C)	If a = 1 and b = 2, then (x, y)	(r)	lies on y = x
(D)	If a = 2 and b = 2, then (x, y)	(s)	lies on $(4x^2 - 1)(y^2 - 1) = 0$

6. If 0 < x< 1, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ [JEE 2008, 3]

(A) $\dfrac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

7. The value of $\cot\left(\displaystyle\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is [JEE (Advanced) 2013, 2]

(A) $\dfrac{23}{25}$ (B) $\dfrac{25}{23}$ (C) $\dfrac{23}{24}$ (D) $\dfrac{24}{23}$

8. Match List-I with List-II and select the correct answer using the code given below the lists.

List-I | **List-II**

P. $\left(\frac{1}{y^2}\left(\frac{\cos\left(\tan^{-1} y\right)+y\sin\left(\tan^{-1} y\right)}{\cot\left(\sin^{-1} y\right)-\tan\left(\sin^{-1} y\right)}\right)^2+y^4\right)^{1/2}$ takes value — 1. $\frac{1}{2}\sqrt{\frac{5}{3}}$

Q. If cosx + cosy + cosz = 0 = sinx + siny + sinz then possible value of $\cos\frac{x-y}{2}$ is — 2. $\sqrt{2}$

R. If $\cos\left(\frac{\pi}{4}-x\right)$ cos2x + sinx sin2x secx = cosx sin2x secx+ $\cos\left(\frac{\pi}{4}+x\right)$ cos2x then possible value of secx is — 3. $\frac{1}{2}$

S. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right)=\sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right), x \neq 0$, then possible value of x is — 4. 1

Codes :

	P	Q	R	S
(A)	4	3	1	2
(B)	4	3	2	1
(C)	3	4	2	1
(D)	3	4	1	2

[JEE-Advanced 2013, 3, (–1)]

PREVIOUS YEARS QUESTIONS — **ANSWER KEY** — EXERCISE-5 [B]

2. D 3. A 4. 1 5. (A) - p, (B) - q, (C) - p, (D) - s 6. C 7. B 8. B